

Mass Degeneracies In Self-Dual Models*

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Abstract

An algebraic restriction of the nonabelian self-dual Chern-Simons-Higgs systems leads to coupled abelian models with interesting mass spectra. The vacua are characterized by embeddings of $SU(2)$ into the gauge algebra, and in the broken phases the gauge and real scalar masses coincide, reflecting the relation of these self-dual models to $N = 2$ SUSY. The masses themselves are related to the exponents of the gauge algebra, and the self-duality equation is a deformation of the classical Toda equations.

The self-dual Chern-Simons systems [1, 2, 3, 4] may be characterized by the fact that the energy density possesses a Bogomol'nyi lower bound which is saturated by solutions to first-order self-duality equations. The special form of the self-dual potential may also be fixed by embedding this bosonic system into a supersymmetric model and imposing the condition of $N = 2$ SUSY [5]. Each of these characterizations is familiar from other self-dual systems [6, 7]. In this Letter we investigate another characterization of self-dual models, in terms of the spectra of massive excitations in the various vacua, when the model is viewed as a spontaneous symmetry breaking system. This is of particular interest for Chern-Simons systems because the Higgs mechanism works in an unfamiliar manner [8]. We find that in self-dual Chern-Simons theories associated with the simply-laced A-D-E Lie algebras

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the gauge and real scalar mass spectra are degenerate in each of the (many) inequivalent vacua. Furthermore, the masses are given by a simple universal mass formula in terms of the exponents of the algebra.

The nonabelian relativistic self-dual Chern-Simons system [1, 2, 3, 4] is described by the following Lagrange density (in $2 + 1$ dimensional spacetime)

$$\mathcal{L} = -\text{tr} \left((D_\mu \phi)^\dagger D^\mu \phi \right) - \kappa \epsilon^{\mu\nu\rho} \text{tr} \left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) - V(\phi, \phi^\dagger) \quad (1)$$

where the gauge invariant scalar field potential $V(\phi, \phi^\dagger)$ is

$$V(\phi, \phi^\dagger) = \frac{1}{4\kappa^2} \text{tr} \left\{ \left([[\phi, \phi^\dagger], \phi] - v^2 \phi \right)^\dagger \left([[\phi, \phi^\dagger], \phi] - v^2 \phi \right) \right\}. \quad (2)$$

The covariant derivative is $D_\mu \equiv \partial_\mu + [A_\mu, \]$, the space-time metric is taken to be $g_{\mu\nu} = \text{diag}(-1, 1, 1)$, and tr refers to the trace in a finite dimensional representation of the compact simple Lie algebra \mathcal{G} to which the gauge fields A_μ and the charged scalar matter fields ϕ and ϕ^\dagger belong. For the particular sixth order potential (2) the energy density of this system has a Bogomol'nyi lower bound which is saturated by solutions to the “*relativistic self-dual Chern-Simons equations*”:

$$D_- \phi = 0 \quad (3)$$

$$F_{+-} = \frac{1}{\kappa^2} [v^2 \phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger] \quad (4)$$

where $D_- \equiv D_1 - iD_2$, and F_{+-} is the gauge curvature. The nonrelativistic limit of this system is obtained by ignoring the quartic ϕ term in (4), in which case we arrive at the nonrelativistic self-dual Chern-Simons equations which are integrable [9]. In the relativistic case the general solution is not known, even in the abelian model [1], although certain properties of the abelian solutions can be obtained from asymptotic analysis and a radial ansatz. The only closed-form solutions known in the relativistic case correspond to the zero energy solutions, for which the gauge field is pure gauge and the ϕ field is gauge equivalent to a solution of the *algebraic* equation:

$$[[\phi, \phi^\dagger], \phi] = v^2 \phi. \quad (5)$$

Solutions of this equation also correspond to the minima of the potential (2), and these potential minima are clearly degenerate [3, 4].

Here we propose to consider a simplified form of this self-dual system in which the fields are algebraically restricted by the following ansatz:

$$\phi = \sum_{a=1}^r \phi^a E_a$$

$$A_\mu = i \sum_{a=1}^r A_\mu^a h_a \quad (6)$$

where r is the rank of the gauge Lie algebra, and the E_a and h_a refer to the simple root step operators and Cartan subalgebra generators (respectively) in a ‘‘Gell-Mann’’ basis (for simplicity of notation, we consider only the simply-laced algebras):

$$\begin{aligned} [h_a, h_b] &= 0 & [E_a, E_{-b}] &= \delta_{ab} \sum_{c=1}^r \alpha_c^{(a)} h_c & [h_a, E_b] &= \alpha_a^{(b)} E_b \\ \text{tr}(E_a E_{-b}) &= \delta_{ab} & \text{tr}(h_a h_b) &= \delta_{ab} & \text{tr}(h_a E_b) &= 0 \end{aligned} \quad (7)$$

Here K_{ab} is the Cartan matrix which encodes the inner products of the simple roots $\vec{\alpha}^{(a)}$ (which have been normalized to each have $(\text{length})^2 = 2$):

$$K_{ab} \equiv \vec{\alpha}^{(a)} \cdot \vec{\alpha}^{(b)} \quad (8)$$

With this algebraic restriction on the fields the original Lagrange density (1) and potential (2) simplify considerably. Since the gauge fields lie in the Cartan subalgebra, the Chern-Simons term in the Lagrange density decomposes into r copies of an abelian Chern-Simons term, leading to

$$\mathcal{L}_{\text{restricted}} = - \sum_{a=1}^r \left| \partial_\mu \phi^a + i \left(\sum_{b=1}^r A_\mu^b \alpha_b^{(a)} \right) \phi^a \right|^2 - \kappa \sum_{a=1}^r \epsilon^{\mu\nu\rho} \partial_\mu A_\nu^a A_\rho^a - V \quad (9)$$

where the potential (2) becomes

$$V_{\text{restricted}} = \frac{v^4}{4\kappa^2} \sum_{a=1}^r |\phi^a|^2 - \frac{v^2}{2\kappa^2} \sum_{a=1}^r \sum_{b=1}^r |\phi^a|^2 K_{ab} |\phi^b|^2 + \frac{1}{4\kappa^2} \sum_{a=1}^r \sum_{b=1}^r \sum_{c=1}^r |\phi^a|^2 K_{ab} |\phi^b|^2 K_{bc} |\phi^c|^2 \quad (10)$$

The Lagrange density (9) describes r abelian Chern-Simons gauge fields A_μ^a coupled to r complex scalar fields ϕ^a , with the couplings determined by the Cartan matrix of the original nonabelian gauge algebra. With this algebraic restriction, the relativistic self-dual Chern-Simons equations (3,4) combine into the single set of coupled equations:

$$\partial_+ \partial_- \ln |\phi^a|^2 = - \frac{v^2}{\kappa^2} \sum_{b=1}^r K_{ab} |\phi^b|^2 + \frac{1}{\kappa^2} \sum_{b=1}^r \sum_{c=1}^r |\phi^b|^2 K_{bc} |\phi^c|^2 K_{ac} \quad (11)$$

In the nonrelativistic limit (with factors of the speed of light c restored [1, 2]), the quartic term on the RHS of (11) drops out and one is left with the classical Toda system, which is known to be integrable and explicitly solvable in terms of r holomorphic functions. In the relativistic case, the self-duality equation (11) is a *deformation* of the Toda system.

In this Letter we treat the restricted self-dual model with Lagrange density (9) as a spontaneous symmetry breaking problem, with the gauge field acquiring masses by the Chern-Simons-Higgs mechanism. The potential has a number of (degenerate) minima, and we shall examine the spectrum of fluctuations about these various vacua.

To investigate the vacuum structure we first identify the zero energy solutions $\phi_{(0)}$. These correspond to the minima of the potential and so correspond to solutions of (5) with the field ϕ restricted as in (6). With this ansatz, the condition (5) becomes

$$\sum_{a=1}^r |\phi_{(0)}^a|^2 \phi_{(0)}^b K_{ba} = v^2 \phi_{(0)}^b \quad (12)$$

When $\phi_{(0)}^a \neq 0$ for all a , the solution is given by

$$|\phi_{(0)}^a|^2 = \sum_{b=1}^r (K^{-1})_{ab} \quad (13)$$

where K^{-1} is the inverse Cartan matrix. This can also be characterized as follows: in the representation theory of Lie algebras [10], an important role is played by the sum $\vec{\rho}$ of all fundamental weights $\vec{\lambda}^{(a)}$, which is also equal to one half times the sum of all positive roots. This sum may be re-expanded in terms of the simple roots $\vec{\alpha}^{(a)}$, and when this is done, the coefficients are equal to one half times the values for $(\phi_{(0)}^a)^2$ given in (13).

$$\vec{\rho} = \sum_{a=1}^r \vec{\lambda}^{(a)} = \frac{1}{2} \sum_{\alpha>0} \vec{\alpha} = \frac{1}{2} \sum_{a=1}^r (\phi_{(0)}^a)^2 \vec{\alpha}^{(a)} \quad (14)$$

Table 1 lists the vector $\vec{\rho}$, expanded in the basis of simple roots $\vec{\alpha}^{(a)}$, for each of the simply-laced algebras. We call this solution (13) the “principal embedding” vacuum, for a reason to be explained below.

Algebra	Coefficients $(\phi_{(0)}^a)^2$ in $\vec{\rho} = (1/2) \sum_a (\phi_{(0)}^a)^2 \vec{\alpha}^{(a)}$							
A_r	r	2(r-1)	3(r-2)	...	3(r-2)	2(r-1)	r	
D_r	2r-2	2(2r-3)	3(2r-4)	...	(r-2)(r+1)	r(r-1)/2	r(r-1)/2	
E_6	16	30	42	30	16	22		
E_7	34	66	96	75	52	27	49	
E_8	58	114	168	220	270	182	92	136

Table 1: The expansion of $\vec{\rho}$, half the sum of positive roots, in the basis of simple roots. The coefficients $(\phi_{(0)}^a)^2$ give the components of the “principal embedding” vacuum solution (13).

Alternatively, one can look for solutions to (12) for which one or more of the coefficients $\phi_{(0)}^a$ is zero. If, for example, $\phi_{(0)}^b = 0$, then equation (12) decouples into two or more equations

for the remaining nonzero coefficients. Effectively this amounts to deleting the b^{th} row and b^{th} column from the Cartan matrix K and solving the decoupled conditions. The deletion of the b^{th} row and b^{th} column from the Cartan matrix can be represented diagrammatically as the deletion of the b^{th} dot from the Dynkin diagram of \mathcal{G} .¹ This deletion divides the Dynkin diagram into two or more Dynkin diagrams for algebras of smaller rank. The general solution to the vacuum condition (12) consists of taking the principal embedding solution (13) for each of these subdiagrams. In $A_r \equiv SU(r+1)$ there are $p(r+1)$ ways of performing these successive deletions, where $p(r+1)$ is the number of (unrestricted) partitions of $r+1$. This gives the total number of inequivalent vacua for the A_r system [4].

We now determine the spectrum of massive excitations in these various vacua. In the *unbroken* vacuum, with $\phi_{(0)} = 0$, there are r complex scalar fields, each with mass

$$m = \frac{v^2}{2\kappa} \quad (15)$$

In one of the broken vacua, with $\phi_{(0)} \neq 0$, some of these $2r$ real massive scalar degrees of freedom are converted to massive gauge degrees of freedom. We shall concentrate initially on the principal embedding vacuum (13). This vacuum has the property that *all* r gauge modes acquire a mass, leaving r massive (real) scalar modes. The new scalar masses are determined by expanding the shifted potential $V(\phi + \phi_{(0)})$ to quadratic order in the field ϕ . For the principal embedding vacuum (13) this leads to:

$$V(\phi + \phi_{(0)}) = \frac{v^4}{\kappa^2} \text{tr} \left(\left| [\phi_{(0)}, [\phi_{(0)}^\dagger, \phi]] \right|^2 \right) = \frac{v^4}{\kappa^2} \sum_{a=1}^r \sum_{b=1}^r \phi^a \left(\phi_{(0)}^a \phi_{(0)}^b \sum_{c=1}^r (\phi_{(0)}^c)^2 K_{ac} K_{bc} \right) \phi^b \quad (16)$$

The real scalar masses are then given by the square roots of the eigenvalues of the $r \times r$ mass (squared) matrix:

$$\mathcal{M}_{ab}^{(\text{scalar})} = 4 m^2 \phi_{(0)}^a \phi_{(0)}^b \sum_{c=1}^r (\phi_{(0)}^c)^2 K_{ac} K_{bc} \quad (17)$$

where the $\phi_{(0)}^a$ are given by (13). The resulting mass spectra are listed in Table 2 for the simply-laced algebras. Note that all the masses are *integer* multiples of the mass scale m in (15).

The gauge masses are determined by expanding $\text{tr} \left(\left| D_\mu (\phi + \phi_{(0)}) \right|^2 \right)$ and extracting the piece quadratic in the gauge field A :

$$v^2 \text{tr} \left([A_\mu, \phi_{(0)}]^\dagger [A^\mu, \phi_{(0)}] \right) \quad (18)$$

¹This is because the Cartan matrix K may be viewed as the connection matrix for the Dynkin diagram in the sense that an off-diagonal entry $K_{ab} = -1$ means that the a^{th} and b^{th} dots in the Dynkin diagram are connected by a single line.

Algebra	Masses							
A_r	2	6	12	20	30	...	$r(r-1)$	$r(r+1)$
D_r	2	12	30	56	90	...	$(2r-3)(2r-2)$	$r(r-1)$
E_6	2	20	30	56	72	132		
E_7	2	30	56	90	132	182	306	
E_8	2	56	132	182	306	380	552	870

Table 2: The scalar masses, in units of m , for the principal embedding vacuum (13), obtained as square roots of the eigenvalues of the mass matrix in (17).

In the principal embedding vacuum, this leads to the following $r \times r$ mass matrix:

$$\mathcal{M}_{ab}^{(\text{gauge})} = 2 m \sum_{c=1}^r (\phi_{(0)}^c)^2 \alpha_a^{(c)} \alpha_b^{(c)} \quad (19)$$

where the $\phi_{(0)}^c$ are given by (13).

Here we stress an important difference between the Chern-Simons-Higgs mechanism and the conventional Higgs mechanism in a Yang-Mills gauge theory. In a Yang-Mills theory, the gauge masses produced by the Higgs mechanism would be the *square roots* of the eigenvalues of the mass matrix obtained from (18). However, for gauge masses produced by the Chern-Simons Higgs mechanism [8], the masses are just the eigenvalues of the mass matrix. This difference is essentially because the Chern-Simons Lagrange density is *first order* in spacetime derivatives. By explicit computation, the eigenvalues of the gauge mass matrix (19) yield a gauge mass spectrum which coincides exactly with the scalar mass spectrum presented in Table 2. In other words, the eigenvalues of (17) are the squares of the eigenvalues of (19).

Thus, the mass spectra of the gauge and scalar modes are *degenerate* in the principal embedding vacuum. Such a degeneracy had been noted in the abelian system, which corresponds to the $A_1 \equiv SU(2)$ case of the present model (9,10). In the abelian model [1] there is only one nontrivial vacuum, and a consequence of the particular 6th order self-dual form of the potential is that in this broken vacuum the massive gauge excitation and the real massive scalar field are degenerate in mass. This was also found to be true for the $SU(N)$ system [4]. This pairing of the masses is a reflection of the fact that the self-dual Chern-Simons system (1,2) (and hence also the system (9,10)) can be embedded into an $N = 2$ supersymmetric model [5]. Such an embedding would be impossible if the bosonic masses did not pair up in each vacuum. The 2 + 1 dimensional Abelian Higgs model, which can also be embedded into an $N = 2$ SUSY theory at the self-dual point, also has the feature that, with the self-dual potential, the gauge and scalar masses in the broken vacuum are degenerate [6].

To explain the algebraic origin of this remarkable mass degeneracy of the system (9,10), and to explain the particular masses that arise, we reconsider the vacuum condition (5).

With a factor of v absorbed into the fields, this can be viewed as an embedding of $SU(2)$ into the original gauge algebra \mathcal{G} .² Thus, the vacuum solution $\phi_{(0)}$ may be identified with an $SU(2)$ raising operator J_+ , and so on. Then the quadratic gauge field term in (18) may be re-cast in terms of the adjoint action of $SU(2)$ on the gauge algebra \mathcal{G} :

$$m \operatorname{tr} (A_\mu (J_+ J_- + J_- J_+) A^\mu) = m \operatorname{tr} (A_\mu (\mathcal{C} - J_3^2) A^\mu) \quad (20)$$

where \mathcal{C} is the $SU(2)$ quadratic Casimir. But $J_3 A^\mu = 0$ since the gauge fields are restricted to the Cartan subalgebra by the ansatz (6). Thus the gauge masses are just given by the eigenvalues of the quadratic Casimir \mathcal{C} in the adjoint action (corresponding to the particular $SU(2)$ embedding) of $SU(2)$ on the gauge algebra \mathcal{G} . It is a classical result of Lie algebra representation theory [12] that the adjoint action of the “principal $SU(2)$ embedding” (13) on \mathcal{G} divides the $d \times d$ dimensional adjoint representation of \mathcal{G} into r irreducible sub-blocks, each of dimension $(2s_a + 1)$ where the s_a are known as the *exponents* of the algebra \mathcal{G} . Here d is the dimension of the algebra \mathcal{G} . This sub-blocking fills out the entire $d \times d$ adjoint representation since the exponents have the property that

$$\sum_{a=1}^r (2s_a + 1) = d \quad (21)$$

The elements of each irreducible sub-block are arranged according to their corresponding *principal grading* which is their J_3 eigenvalue. Restricting to the Cartan subalgebra (as is achieved by the ansatz (6) for the gauge fields) selects the $j_3 = 0$ element from each sub-block, and in each irreducible sub-block the quadratic Casimir \mathcal{C} has eigenvalue $\mathcal{C} = s_a(s_a + 1)$. The exponents for the classical simply-laced Lie algebras are listed in Table 3. It is straightforward to verify that the mass spectrum in Table 2 for the eigenvalues of the gauge mass matrix (19) coincides with the general mass formula

$$m_a = m s_a(s_a + 1) \quad a = 1, \dots, r \quad (22)$$

where the s_a are the *exponents* of \mathcal{G} .

To see that the real scalar masses are also given by the general mass formula (22), we note that the quadratic part (16) of the shifted scalar potential can be written as

$$4m^2 \operatorname{tr} (\phi^\dagger (J_+ J_-)^2 \phi) = m^2 \operatorname{tr} (\phi^\dagger (\mathcal{C} - J_3^2 + J_3)^2 \phi) \quad (23)$$

But $J_3 \phi = 1\phi$ since ϕ is expanded in terms of the simple root step operators (and hence has principal grading 1). Thus the eigenvalues of the scalar $mass^2$ matrix are the squares of the

²It is interesting to note that this type of embedding problem also plays a significant role in the theory of instantons and of spherically symmetric magnetic monopoles and the Toda molecule equations [11].

Algebra	Rank	Dimension	Exponents						
A_r	r	$r(r+2)$	1	2	3	\dots	$r-1$	r	
D_r	r	$r(2r-1)$	1	3	5	\dots	$2r-3$	$r-1$	
E_6	6	78	1	4	5	7	8	11	
E_7	7	133	1	5	7	9	11	13	17
E_8	8	248	1	7	11	13	17	19	23 29

Table 3: The ranks, dimensions and exponents of the simply-laced classical Lie algebras. Note that the sum of the exponents equals the number of positive roots, which is one half (dimension – rank).

eigenvalues of \mathcal{C} , and we find a scalar mass spectrum identical with the gauge mass spectrum in (22).

For any vacuum $\phi_{(0)}$ *other* than the principal embedding one (13), the gauge and scalar masses may be found as follows. If the vacuum solution $\phi_{(0)}$ corresponds to n deletions of dots from the original Dynkin diagram (as described before - see also [4]) then n complex scalar fields remain massive with mass m corresponding to the scalar mass in the unbroken vacuum. The remaining $(r - n)$ real scalar masses are obtained from formula (22) using the exponents for each of the Dynkin sub-diagrams. This also yields the $(r - n)$ real gauge masses. Thus in any vacuum, the masses are always paired, either because they correspond to a complex scalar field (of which the extreme case is the unbroken vacuum) or because the real scalar and gauge masses coincide through formula (22) (of which the principal embedding vacuum (13) is the extreme case).

To conclude, we have shown that in the coupled self-dual Chern-Simons systems (9,10) the gauge and real scalar masses in the broken vacua are degenerate and are given by the universal mass formula (22) in terms of the exponents of the associated Lie algebra \mathcal{G} . It is interesting to note that in the affine Toda systems, the masses are also related to the exponents of the associated gauge algebra [13]. The quantum implications, associated with tunnelling effects for example, of these mass degeneracies and mass spectra remain to be resolved.

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